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INITIAL VALUE PROBLEMS IN VISCOELASTICITY

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INITIAL VALUE PROBLEMS IN VISCOELASTICITY

W. J. Hrusa*, J. A. Nohel**, and M. Renardy***

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Abstract

~~We~~ review some recent mathematical results concerning integrodifferential equations that model the motion of one-dimensional nonlinear viscoelastic materials. In particular, ~~we~~ discuss global (in time) existence and long-time behavior of classical solutions, as well as the formation of singularities in finite time from smooth initial data. Although the mathematical theory is comparatively incomplete, ~~we make~~ some remarks concerning the existence of weak solutions (i.e., solutions with shocks). Some relevant results from linear wave propagation will also be discussed.

AMS (MOS) Subject Classifications: 35A22, 35L15, 35L67, 35L70, 35R05, 45E10, 45K05, 73D10, 73F15, 73F99, 73H99, 76A10.

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INITIAL VALUE PROBLEMS IN VISCOELASTICITY

W. J. Hrusa*, J. A. Nohel**, and M. Renardy***

1. Introduction

Over the last decade or so, a significant amount of research has been devoted to the study of integrodifferential equations which model the motion of nonlinear viscoelastic materials. The purpose of this paper is to review some of the results that have been obtained and some problems that remain open. We focus on equations that describe viscoelastic solids; a similar paper concerning viscoelastic fluids will appear elsewhere.

Our presentation will be informal in the sense that technical hypotheses of smoothness, etc. will not always be stated explicitly, and we shall not attempt to state results in the greatest possible generality. We refer to the original sources and to our recent monograph (Renardy, Hrusa, and Nohel (1987), henceforth referred to as RHN) for precise statements of technical assumptions and for more general results. In order to avoid geometrical complications, and because the mathematical theory of such equations is far more complete, we restrict our attention to problems involving a single spatial coordinate. Moreover, we always treat the full dynamical equations with inertia; quasistatic approximations will not be discussed.

Consider the longitudinal motion of a homogeneous one-dimensional body, e.g. a bar of uniform cross-section. [This problem is somewhat artificial since a longitudinally deforming bar may experience a variation in its cross-section. There are, however, more realistic classes of motions that can be described with a single spatial coordinate (cf., e.g.,

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the appendix of Coleman, Gurtin, and Herrera (1965) and Section I.4 of RHN).] We assume that the body occupies an interval $B \subset \mathbb{R}$ in an unstressed state. The interval B will be called the reference configuration; a typical point (or particle) in B will be denoted by x. In order to describe the motion, we follow the evolution of points in B. Therefore, the quantities of interest will be regarded as functions of the material point x and the time t.

We assume that the body has unit reference density, and we denote by $u(x,t)$ the displacement at time t of the particle with reference position x (i.e., $x + u(x,t)$ is the position at time t of this particle). The strain ϵ is given by

$$\epsilon(x,t) = u_x(x,t), \tag{1.1}$$

and the equation of balance of linear momentum has the form

$$u_{tt}(x,t) = \sigma_x(x,t) + f(x,t), \qquad x \in B, \quad t \geq 0, \tag{1.2}$$

where σ is the stress and f is a (prescribed) body force. Subscripts x and t indicate partial derivatives; we use a prime to denote the derivative of a function of one variable. The type of material composing the body is characterized by a constitutive assumption which relates the stress to the motion.

If the body is elastic then the stress depends on the strain through a constitutive equation of the form

$$\sigma(x,t) = \varphi(\epsilon(x,t)), \tag{1.3}$$

where φ is a given smooth function with $\varphi(0) = 0$ and $\varphi'(0) > 0$. The condition $\varphi(0) = 0$ reflects the assumption that reference configuration be stress-free, while the condition $\varphi'(0) > 0$ means that stress increases with strain, at least near $\epsilon = 0$. For simplicity, let us assume temporarily that $B = \mathbb{R}$ and $f \equiv 0$. The resulting motion is then governed by the quasilinear wave equation

$$u_{tt} = \varphi(u_x)_x, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.4}$$

We seek a function u that satisfies (1.4) together with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}. \tag{1.5}$$

In the linear case, $\sigma = E\epsilon$ (with E a positive constant), equation (1.4) reduces to

$$u_{tt} = Eu_{xx}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.6}$$

A well-known feature of (1.6) is that singularities in the data (i.e. discontinuities in the data or in derivatives of the data) persist as singularities in the solution for all $t > 0$ and they propagate with constant speed $\sqrt{E}$. The amplitude of such a singularity neither grows nor decays.

For the linear equation (1.6), singularities will be present in the solution if and only if they are present in the data. In the nonlinear case the solution generally develops singularities (in finite time) – no matter how smooth the data are – because the wave speed $\sqrt{\varphi'}$ is not constant. This issue will be discussed in more detail in Section 4.

Experience indicates that certain materials have memory, i.e. the stress depends not only on the strain at the present time, but also on the entire temporal history of the strain. Typically the memory fades with time, i.e. disturbances which occurred in the distant past have less influence on the present stress than those which occurred in the more recent past. We shall refer to such materials as viscoelastic. The linearized constitutive relation for small deformations was given by Boltzmann (1876):

$$\sigma(x,t) = \beta\epsilon(x,t) + \int_{-\infty}^{t} m(t-\tau)(\epsilon(x,t) - \epsilon(x,\tau))d\tau. \tag{1.7}$$

Here β is a non-negative constant, and m is a positive monotone-decreasing function. To ensure that (1.7) is meaningful, we assume that m is integrable at infinity and that $sm(s)$ is integrable at zero, i.e.

$$\int_{1}^{\infty} m(s)ds < \infty, \quad \int_{0}^{1} sm(s)ds < \infty. \tag{1.8}$$

If m is integrable on $(0, \infty)$, i.e.

$$\int_0^\infty m(s)ds < \infty, \tag{1.9}$$

we can rewrite (1.7) in the alternative form

$$\sigma(x,t) = E\epsilon(x,t) - \int_{-\infty}^t m(t-\tau)\epsilon(x,\tau)d\tau, \tag{1.10}$$

where

$$E = \beta + \int_0^\infty m(s)ds. \tag{1.11}$$

(Since m is assumed to be positive, decreasing, and integrable at infinity, (1.9) holds if and only if m is integrable at zero.) The constant E measures the instantaneous response of stress to strain and is called the instantaneous stress modulus. On the other hand, β determines the stress due to a constant strain history and is called the equilibrium stress modulus; for viscoelastic solids it is natural to assume $\beta > 0$. If we set $\epsilon = 0$ for $t < 0$ and $\epsilon = \epsilon_0$ for $t > 0$, then the stress for $t > 0$ is given by

$$\sigma = (\beta + \int_t^\infty m(s)ds)\epsilon_0 = G(t)\epsilon_0. \tag{1.12}$$

The function G defined through (1.12) is called the stress relaxation modulus. An experiment in which the strain is suddenly changed from 0 and held at a constant value is called a stress relaxation test; by measuring the stress one obtains $G(t)$. Obviously, $G(0) = E$ and $G(\infty) = \beta$. The function $m(t) = -G'(t)$ is called the memory function. The assumption that m is positive means that G is monotone-decreasing, i.e. that the stress following a sudden deformation relaxes with increasing time. It follows from (1.7) and the positivity of m that the stress needed to sustain a strain ϵ is diminished by previous strains of the same sign and increased by previous strains of the opposite sign. The assumption that m is monotone-decreasing (i.e., that G is convex) means that a deformation which occurred in the distant past has less influence on the present stress than one which occurred in the more recent past, i.e. the memory fades with time.

Substitution of (1.7) into balance of linear momentum (1.2) yields

$$u_{tt}(x,t) = \beta u_{xx}(x,t) + \int_{-\infty}^{t} m(t-\tau)(u_{xx}(x,t) - u_{xx}(x,\tau))d\tau + f(x,t) \tag{1.13}$$

or

$$u_{tt}(x,t) = Eu_{xx}(x,t) - \int_{-\infty}^{t} m(t-\tau)u_{xx}(x,\tau)d\tau + f(x,t), \tag{1.14}$$

where the second form is valid only if m is integrable. We assume that the history of u prior to time $t = 0$ as well as the values of $u(x,0)$ and $u_t(x,0)$ are prescribed.

In much of the work on viscoelasticity it is assumed that m is smooth on $[0,\infty)$ and hence that $m(0)$ is finite. There are, however, theoretical and experimental indications that for certain viscoelastic materials it is reasonable to assume

$$m(0) = +\infty \tag{1.15}$$

(cf., e.g. Doi and Edwards (1978, 1979), Joseph, Riccius, and Arney (1986), Laun (1978), Rouse (1953), and Zimm (1956)). Memory functions satisfying (1.15) will be called singular. We always assume, without explicit mention, that m is smooth on $(0,\infty)$ and that (1.8) holds. However, we shall always state explicitly whether or not m is required to be smooth on $[0,\infty)$. For singular memory functions, an important distinction arises as to whether or not the singularity is integrable, i.e. whether or not m is integrable at zero. We note that if m has a nonintegrable singularity, e.g. $m(t) = e^{-t}t^{-3/2}$, then $G(0) = +\infty$.

Singular memory functions have been employed in applications to many important engineering problems (cf., e.g. Bert (1973) and Walton, Nachman and Schapery (1978)). The possibility of a singular memory function was already considered by Boltzmann (1876) who proposed that $m(s) \sim s^{-1}$ (except for large s) and fitted such a kernel to torsion measurements on a filament of glass.

Nonlinear viscoelastic materials are characterized by constitutive equations which express the stress as a (nonlinear) functional of the temporal history of the strain. We consider only materials which are simple (cf. Noll (1958)) in the sense that the stress at a material point depends only on the history of the strain at that same material point (and not on the histories of the strain at other material points or on spatial derivatives of the strain). We suppress the variable x in our discussion of constitutive equations.

General constitutive theories for nonlinear viscoelastic behavior which do not assume a specific form for the stress functional have been given by Coleman and Noll (1960), Coleman and Mizel (1967, 1968), Saut and Joseph (1982) and Wang (1965). We shall restrict our attention to constitutive relations of single-integral type with one memory function:

$$\sigma(t) = F(\epsilon(t)) + \int_{-\infty}^{t} m(t-\tau)M(\epsilon(t), \epsilon(\tau))d\tau. \tag{1.16}$$

Here F and M are given smooth functions with $F(0) = 0$, $M(p,p) = 0$ for all p, and m is as above. The function F measures the stress due to a strain history that is constant in time; its derivative F' is called the equilibrium stress modulus. For viscoelastic solids it is natural to assume that $F'(\epsilon) > 0$, at least for ϵ near zero.

If m is integrable we may rewrite (1.16) in the form

$$\sigma(t) = \int_{-\infty}^{t} m(t-\tau)H(\epsilon(t), \epsilon(\tau))d\tau, \tag{1.17}$$

where

$$H(\epsilon, p) = M(\epsilon, p) + F(\epsilon)\left(\int_0^{\infty} m(s)ds\right)^{-1}. \tag{1.18}$$

If we fix t and hold the values of $\epsilon(\tau)$ fixed for $-\infty < \tau < t$, but vary the present value $\epsilon(t)$, then the integral in (1.17) can be regarded as a function of $\epsilon(t)$. The derivative of this function is called the instantaneous elastic modulus; it measures the change in stress due to an instantaneous change in strain. We note that the instantaneous elastic modulus depends on the present value of the strain as well as the entire past history of the strain. The equilibrium stress modulus for (1.17) is given by

$$[H_{,1}(\epsilon, \epsilon) + H_{,2}(\epsilon, \epsilon)]\int_0^{\infty} m(s)ds, \tag{1.19}$$

where $H_{,1}$ and $H_{,2}$ denote the derivative of H with respect to its first and second argument, respectively.

The special case of (1.17) that occurs when

$$H(\epsilon, p) = f(\epsilon) + g(p) \tag{1.20}$$

will be considered in Sections 5 and 6. In order to be consistent with the literature on this model we introduce the notations

$$\varphi(\epsilon) = f(\epsilon) \int_0^\infty m(s)ds, \quad \psi(p) = -g(p), \tag{1.21}$$

and write the constitutive relation in the form

$$\sigma(t) = \varphi(\epsilon(t)) - \int_{-\infty}^t m(t-\tau)\psi(\epsilon(\tau))d\tau. \tag{1.22}$$

We note that for (1.22), the instantaneous elastic modulus depends only on the present value of the strain and is given by $\varphi'(\epsilon)$; the equilibrium stress modulus is given by

$$\varphi'(\epsilon) - \psi'(\epsilon) \int_0^\infty m(s)ds. \tag{1.23}$$

For the constitutive relation (1.17) the equation of balance of linear momentum (1.2) becomes

$$u_{tt}(x,t) = \int_{-\infty}^t m(t-\tau)H(u_x(x,t), u_x(x,\tau))_x d\tau + f(x,t), \qquad x \in B, \quad t \geq 0. \tag{1.24}$$

We assume that the history of u prior to time $t = 0$ as well as the values of $u(x,0)$ and $u_t(x,0)$ are prescribed, i.e. we take initial data of the form

$$u(x,\tau) = w(x,\tau), \qquad x \in B, \quad \tau < 0, \tag{1.25}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.26}$$

where w, u_0, and u_1 given functions. Unless specified otherwise, we assume that w, u_0, and u_1 are smooth. However, we do not necessarily assume that u_0 and u_1 are compatible with w in the sense that $u_0(x) = w(x,0^-)$, $u_1(x) = w_t(x,0^-)$. If the interval B is finite (or semi-infinite) then boundary conditions will be imposed at the endpoints of B; typically, either the displacement or the stress will be prescribed at each endpoint.

Carrying out the differentiation in (1.24) we obtain

$$
\begin{aligned}
u_{tt}(x,t) = {} & \left(\int_{-\infty}^{t} m(t-\tau)H_{,1}(u_x(x,t),u_x(x,\tau))d\tau\right)u_{xx}(x,t) \\
& + \int_{-\infty}^{t} m(t-\tau)H_{,2}(u_x(x,t),u_x(x,\tau))u_{xx}(x,\tau)d\tau \\
& + f(x,t), \quad x \in B, \quad t \geq 0. \qquad (1.27)
\end{aligned}
$$

Observe that the coefficient of $u_{xx}(x,t)$ is precisely the instantaneous elastic modulus.

The paper is organized as follows. Section 2 is concerned with linear wave propagation. Our purpose there is to discuss the effects of memory on a discontinuity in the data, with particular emphasis on the effects of a singular kernel. The remainder of the paper is devoted to nonlinear problems.

Section 3 contains a brief discussion of acceleration waves. Section 4 is concerned with the formation of singularities in solutions with smooth (but large) initial data. In Section 5 we discuss the global existence and asymptotic behavior of classical solutions for smooth and small data.

Comparatively little is known about the existence of weak solutions (i.e. solutions with shocks); this topic is currently under active investigation. Some remarks on these matters will be given in Section 6.

2. Linear Wave Propagation

In this section we give a brief synopsis of some results concerning linear wave propagation. There is a very large literature on this subject (cf., e.g. the monograph of Christensen (1971) for references to early work). We refer to Section II.3 of RHN for a much more extensive discussion of the results quoted here.

We concentrate on the so-called Rayleigh problem, in which a discontinuity is introduced at the boundary of a semi-infinite linearly viscoelastic body:

$$u_{tt}(x,t) = \beta u_{xx}(x,t) + \int_{-\infty}^{t} m(t-\tau)(u_{xx}(x,t) - u_{xx}(x,\tau))d\tau, \quad x \geq 0, \quad t \geq 0, \tag{2.1$_1$}$$

$$u(0,t) = 1, \qquad t > 0, \tag{2.1$_2$}$$

$$u(x,\tau) = 0, \qquad x \geq 0, \quad \tau < 0, \tag{2.1$_3$}$$

$$u(x,0) = u_t(x,0) = 0, \qquad x \geq 0. \tag{2.1$_4$}$$

Observe that there is a discontinuity in the data at $x = t = 0$. The solution of (2.1) provides a great deal of information concerning the effects of memory.

To simplify the exposition we assume that m is of class C^{∞} (i.e. infinitely differentiable) on $(0,\infty)$ and that

$$(-1)^k m^{(k)}(t) \geq 0 \qquad \forall t > 0, \quad k = 0,1,\ldots, \tag{2.2}$$

where $m^{(k)}$ is the k-th derivative of m. We note that most of the results quoted below remain valid under much weaker assumptions. However, (2.2) is very reasonable from the viewpoint of applications. Memory functions satisfying this condition are called completely monotonic. (This class of memory functions is very popular in rheology.) We also assume that

$$\beta \geq 0, \quad m \not\equiv 0. \tag{2.3}$$

The behavior of the solution of (2.1) depends on the behavior of m at zero. The case when m has an integrable singularity is especially interesting because smoothing of the discontinuity can coexist with finite speed of propagation.

If m is smooth on $[0,\infty)$ then u is smooth (in fact analytic) for $0 \leq x < ct$, and $u = 0$ for $x > ct$, where

$$c = (\beta + \int_0^{\infty} m(s)ds)^{1/2} \tag{2.4}$$

(cf. Berry (1958)). Across the line $x = ct$, u sustains a jump discontinuity; the amplitude A of the jump is given by

$$A(t) = \exp[-m(0)t/2c^{3/2}] \tag{2.5}$$

(cf. Chu (1962)). In other words the discontinuity at $x = t = 0$ propagates with constant speed and its amplitude decays exponentially. (For a purely elastic material the discontinuity would propagate with constant speed and its amplitude would remain constant.)

If we formally put $m(0) = +\infty$ in (2.5) then $A(t)$ becomes zero for $t > 0$, which suggests that a singularity in m has a smoothing effect on the solution. This is indeed the case; the precise degree of smoothing depends crucially on the strength of the singularity in m. Explicit examples of singular memory functions which lead to varying degrees of smoothing in the solution were given by Renardy (1982) and Hrusa and Renardy (1985).

If m is integrable then u is analytic in the region $0 \leq x < ct$ and $u = 0$ for $x > ct$, where c is given by (2.4). Prüss (1987) has shown that u is continuous across the line $x = ct$ (and hence continuous on the quarter plane $x \geq 0$, $t > 0$) if and only if $m(0) = +\infty$. Desch and Grimmer (1988) showed that u is of class C^∞ on the quarter plane $x \geq 0$, $t > 0$ if and only if m has a singularity which is stronger than logarithmic. They also showed that if m has a singularity which is weaker than logarithmic then u fails to be of class C^1 across the line $x = ct$. An example of a memory function with a logarithmic singularity is given in Hrusa and Renardy (1985); it is shown that the smoothness of u across the line $x = ct$ increases in a manner proportional to t.

If m has a nonintegrable singularity at zero then the solution of (2.1) is analytic in the quarter plane $x \geq 0$, $t > 0$. Since u is analytic, the speed of propagation is infinite.

To summarize, if m is smooth on $[0, \infty)$ then the discontinuity at $x = t = 0$ persists for all $t > 0$, but its amplitude decays exponentially. A singularity in m has a smoothing effect on the discontinuity; the degree of smoothing can range from infinitesimal (u is continuous, but not C^1) all the way to analytic smoothing. The degree of smoothing increases with the strength of the singularity in m.

3. Nonlinearity and Acceleration Waves

The results on linear wave propagation indicate that memory has a damping effect. In the absence of memory, a nonlinear elastic response can lead to the formation of singularities in finite time from arbitrarily smooth initial data; formation of singularities generally occurs even if the data are small. Equation (1.24) includes a nonlinear instantaneous elastic response in conjunction with damping due to memory. A great deal of insight concerning the interaction of a nonlinear instantaneous response with the damping effects of memory is provided by the work of Coleman and Gurtin (1965) on the growth and decay of acceleration waves.

We consider equation (1.24) with $B = \mathbb{R}$ and $f \equiv 0$, i.e.

$$u_{tt}(x,t) = \int_{-\infty}^{t} m(t-\tau)H(u_x(x,t), u_x(x,\tau))_x d\tau, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{3.1}$$

(The analysis of Coleman and Gurtin was carried out for a much more general class of constitutive equations.) Throughout this section we assume that m is smooth on $[0,\infty)$ and that $m > 0$, $H_{,1} > 0$. We consider a solution of (3.1) that is of class C^2 on each side of a smooth curve $t = \lambda(x)$, across which u, u_x, and u_t are continuous, but the second derivatives experience a jump. Such a singularity in the solution is called an acceleration wave. We suppose that an acceleration wave is propagating into an undeformed medium at rest; we assume that $u(x,t) = 0$ for $t < \lambda(x)$. Coleman, Gurtin and Herrera (1965) showed that $\lambda'(x) = c^{-1}$, where

$$c = (H_{,1}(0,0) \int_0^\infty m(s)ds)^{1/2}, \tag{3.2}$$

i.e. the acceleration wave propagates with constant speed. (This conclusion rests heavily on the assumption that the wave is moving into an undeformed medium at rest.)

Let us denote by $A(t)$ the jump in u_{tt} across the curve $t = \lambda(x)$. Coleman and Gurtin showed that A satisfies the ordinary differential equation

$$\frac{dA}{dt} = \alpha A^2 - \beta A, \tag{3.3}$$

where

$$\alpha = -\frac{1}{2c^3}\{H_{,11}(0,0)\int_0^\infty m(s)ds\}, \tag{3.4}$$

$$\beta = \frac{-m(0)}{2c^2}H_{,2}(0,0). \tag{3.5}$$

The differential equation (3.3) can be solved explicitly. The following two consequences of (3.3) are of particular interest; we assume that $\alpha \neq 0$ and $\beta > 0$:

(i) If $|A(0)| < \frac{\beta}{|\alpha|}$ then

$$A(t) \to 0 \text{ as } t \to \infty.$$

(ii) If $|A(0)| > \frac{\beta}{|\alpha|}$ and $\alpha A(0) > 0$ then $A(t)$ becomes infinite in finite time.

The above results suggest that the damping effect of memory will dominate in equation (3.1) if the data are small, while nonlinearity in the instantaneous elastic response will dominate when the data are large. A number of recent results along these lines will be discussed in the next two sections.

We note that for a purely elastic material $H_{,2} \equiv 0$ and hence $\beta = 0$. It follows from equation (3.3) that $A(t)$ becomes infinite in finite time if $\alpha A(0) > 0$. Hence acceleration waves of arbitrarily small initial amplitude can blow up in finite time.

An explicit example of an acceleration wave in a nonlinear viscoelastic material was given by Pipkin (1966). Subsequently, Greenberg (1967) established the existence of solutions with steady acceleration waves for a very general class of viscoelastic materials.

The results of Coleman and Gurtin require that m is smooth on $[0,\infty)$. In the linear case singular kernels lead to instantaneous damping of discontinuities. It is therefore natural to expect that singular kernels may also lead to a much stronger type of damping in nonlinear problems. However, no results of this nature have been obtained so far.

4. Formation of Singularities

If m is smooth on $[0,\infty)$ and $m > 0$, $H_{,1} > 0$ then a relatively straightforward iteration procedure can be used to establish the local (in time) existence of a classical solution to

(1.24) provided the given data are sufficiently smooth. The solution will exist up to a maximal time $T_{max} \leq \infty$; if $T_{max} < \infty$ then second derivatives of u will become infinite in finite time.

The work of Coleman and Gurtin (1965) on acceleration waves suggests that if the data are sufficiently large then T_{max} will be finite, i.e. the local solution will develop singularities in finite time. This is indeed the case; several results of this nature have been established over the last few years.

For purposes of comparison, we begin by reviewing a classical result of Lax (1964) concerning the quasilinear wave equation. Consider the initial-boundary value problem

$$u_{tt} = \varphi(u_x)_x, \qquad 0 \leq x \leq 1, \quad t \geq 0, \tag{4.1$_1$}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{4.1$_2$}$$

$$u(x,0) = 0, \quad u_t(x,0) = u_1(x), \tag{4.1$_3$}$$

where $\varphi' > 0$, $\varphi'' > 0$, and u_1 is smooth and vanishes at the endpoints $x = 0, 1$. (We take $u(x,0) \equiv 0$ only for the sake of simplicity; Lax's argument applies to general initial data.) The problem (4.1) has a unique classical solution on a maximal time interval $[0, T_{max})$. Lax shows that if $u_1(x) \not\equiv 0$ then $T_{max} < \infty$, i.e. the solution develops singularities in finite time unless u_1 vanishes identically (in which case the solution also vanishes identically). As $t \uparrow T_{max}$, the second derivatives of u become infinite. If $u_1(x) \not\equiv 0$ then

$$A = \max_{0 \leq x \leq 1} u_1'(x) > 0 \tag{4.2}$$

since $u_1(0) = u_1(1) = 0$. An approximation for T_{max} which is valid when $\max\limits_{0 \leq x \leq 1} |u_1(x)|$ is sufficiently small is given by

$$T_{max} \approx 4\varphi'(0)(A\varphi''(0))^{-1}. \tag{4.3}$$

Under the above assumptions on φ and the boundary conditions (4.1)$_2$, the result for general initial data is as follows: The solution will develop singularities in finite time unless the initial data for u and u_t both vanish identically. (For general data the analog of (4.3) is more complicated.) The situation is similar if the assumption $\varphi'' > 0$ is replaced by $\varphi'' < 0$.

Lax's argument makes crucial use of the fact that, for $0 \leq x \leq 1$ and $0 \leq t < T_{max}$, the quantity $\varphi''(u_x(x,t))$ never vanishes. Results establishing formation of singularities for equation $(4.1)_1$ when φ has inflection points have been given by MacCamy and Mizel (1967) and Klainerman and Majda (1981).

Under appropriate sign assumptions, the memory in (1.24) induces a dissipative mechanism which competes with the destabilizing effects of a nonlinear instantaneous elastic response. The dissipation dominates if the solution is small, while the elastic response dominates if the solution is large. One can still establish formation of singularities in finite time for solutions of (1.24), but it must be assumed that the data are large enough so that the instantaneous response dominates. The following result is a special case of a theorem of Dafermos (1986).

Consider the problem

$$u_{tt}(x,t) = \int_{-\infty}^{t} m(t-\tau) H(u_x(x,t), u_x(x,\tau))_x d\tau, \quad 0 \leq x \leq 1, \quad t \geq 0, \tag{4.4$_1$}$$

$$u(0,t) = u(1,t) = 0, \tag{4.4$_2$}$$

$$u(x,t) = 0, \qquad 0 \leq x \leq 1, \quad t < 0, \tag{4.4$_3$}$$

$$u(x,0) = 0, \quad u_t(x,0) = u_1(x), \tag{4.4$_4$}$$

where u_1 is smooth and vanishes at the endpoints $x = 0, 1$. We assume that m is smooth on $[0, \infty)$, $m > 0$, $H_{,1} > 0$, and $H_{,11}(0,0) > 0$. The problem (4.4) has a unique classical solution which exists up to a maximal time $T_{max} \leq \infty$. Dafermos proved that given any numbers $N, T > 0$ there are corresponding numbers $\delta, M > 0$ such that if

$$\max_{0 \leq x \leq 1} |u_1(x)| < \delta, \quad \max_{0 \leq x \leq 1} u_1'(x) > M, \quad \min_{0 \leq x \leq 1} u_1'(x) > -N$$

then $T_{max} \leq T$. As $t \uparrow T_{max}$ the second derivatives of u become infinite. Results of a similar nature have been obtained by Kosinski (1977), Slemrod (1978), Gripenberg (1982), Hattori (1982), Nohel and Renardy (1987) and Rammaha (1987); for some numerical work concerning the formation of singularities see Markowich and Renardy (1984).

All of the results concerning formation of singularities for equation $(4.4)_1$ require that $m(s)$ remains bounded as $s \downarrow 0$. It would be very interesting to know whether or not singularities can form from smooth data for nonlinear problems with singular kernels. This question is currently under investigation.

5. Global Existence

This section is concerned with the global existence and asymptotic behavior (as $t \to \infty$) of smooth solutions. We assume throughout that m satisfies

$$m(t) \geq 0, \quad m'(t) \leq 0, \qquad \forall t > 0, \tag{5.1$_1$}$$

$$m \not\equiv 0. \tag{5.1$_2$}$$

Much of the work on global existence has been carried out for the constitutive relation (1.22), i.e. for the equation of motion

$$u_{tt}(x,t) = \varphi(u_x(x,t))_x - \int_{-\infty}^{t} m(t-\tau)\psi(u_x(x,\tau))_x d\tau + f(x,t). \tag{5.2}$$

We assume that u vanishes identically prior to time $t = 0$, and we impose nontrivial data for u and u_t at $t = 0$. Thus we consider initial value problems of the form

$$u_{tt}(x,t) = \varphi(u_x(x,t))_x - \int_{0}^{t} m(t-\tau)\psi(u_x(x,\tau))_x d\tau + f(x,t), \quad x \in B,\ t \geq 0, \tag{5.3$_1$}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in B, \tag{5.3$_2$}$$

together with appropriate boundary conditions. We assume that

$$\int_0^{\infty} m(t)dt < \infty,\ \int_0^{\infty} tm(t)dt < \infty \tag{5.4}$$

$$\varphi(0) = \psi(0) = 0 \tag{5.5$_1$}$$

$$\varphi'(0) > 0, \quad \psi'(0) > 0, \quad \varphi'(0) - \left(\int_0^\infty m(t)dt\right)\psi'(0) > 0. \tag{5.5$_2$}$$

Note that our assumptions permit the kernel m to have an integrable singularity at zero.

In order to obtain global existence we require u_0, u_1, and f to be small and compatible with the boundary conditions. If m is smooth on $[0, \infty)$ then the results of the preceding section indicate that we must restrict the size of the data in order to avoid the formation of singularities in finite time. For $m(0) = +\infty$, it is not known whether one can obtain global existence for arbitrarily large data.

We begin with the case $B = [0, 1]$. To measure the size of the data we define

$$U(u_0, u_1) = \int_0^1 [u_0(x)^2 + u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 + u_1'(x)^2 + u_1''(x)^2]dx, \tag{5.6$_1$}$$

$$F(f) = \int_0^\infty \int_0^1 \{f^2 + f_x^2 + f_t^2 + f_{xt}^2 + f_{tt}^2\}(x,t)dxdt. \tag{5.6$_2$}$$

For the Dirichlet boundary conditions

$$u(0,t) = u(1,0) = 0, \qquad t \geq 0 \tag{5.7}$$

the compatibility conditions will be satisfied if we assume

$$u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0, \tag{5.8$_1$}$$

$$u_0''(0) = u_0''(1) = f(0,0) = f(1,0) = 0. \tag{5.8$_2$}$$

We note that if m is smooth on $[0, \infty)$ then $(5.8)_2$ can be weakened (cf. Theorem IV.5 of RHN). The existence theorem reads as follows: There exists a constant $\mu > 0$ such that if u_0, u_1, and f satisfy (5.8) and

$$U(u_0, u_1) + F(f) \leq \mu, \tag{5.9}$$

then the initial boundary value problem (5.3), (5.7) (with $B = [0, 1]$) has a unique smooth solution u that exists for all $t \geq 0$. Moreover, as $t \to \infty$, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt} converge to zero uniformly in x.

A very similar result holds for the Neumann conditions

$$u_x(0,t) = u_x(1,t) = 0, \qquad t \geq 0. \tag{5.10}$$

(We note that (5.10) is equivalent to $\sigma(0,t) = \sigma(1,t) = 0$, $t \geq 0$; cf., e.g. Section 3 of Dafermos and Nohel (1981).) The boundary conditions (5.10) permit nontrivial rigid motions; in order to eliminate the possibility of such motions we normalize the data so that

$$\int_0^1 u_0(x)dx = \int_0^1 u_1(x)dx = 0 \qquad (5.11)_1$$

$$\int_0^1 f(x,t)dx = 0, \quad \forall t \geq 0. \qquad (5.11)_2$$

There is no loss of generality in assuming (5.11) because a problem with general data can always be converted to an equivalent problem in which the data satisfy (5.11) (cf., e.g. Section 1 of Dafermos and Nohel (1981)). In place of (5.8) we now assume

$$u_0'(0) = u_0'(1) = u_1'(0) = u_1'(1) = 0, \tag{5.12}$$

and the existence theorem reads: There is a constant $\mu > 0$ such that if u_0, u_1, f satisfy (5.9), (5.11), (5.12) then the initial-boundary value problem (5.3), (5.10) has a unique smooth solution u that exists for all $t \geq 0$. Moreover, as $t \to \infty$, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt} converge to zero uniformly in x.

An analogous result holds if (5.10) is replaced by

$$u_x(0,t) = u(1,t) = 0, \quad t \geq 0, \tag{5.13}$$

(or $u(0,t) = u_x(1,t) = 0$). Since the boundary conditions (5.13) do not permit nontrivial rigid motions there is no reason to require (5.11). A suitable "mixed" version of (5.8), (5.12) should be assumed.

For the case when $B = \mathbb{R}$ we introduce

$$U_0^*(u_0,u_1) = \int_{-\infty}^{\infty} [u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 + u_1'(x)^2 + u_1''(x)^2]dx \qquad (5.14)_1$$

$$F^*(f) = \left(\int_0^\infty \left(\int_{-\infty}^\infty f(x,t)^2 dx\right)^{1/2} dt\right)^2 + \int_0^\infty \int_{-\infty}^\infty \{f_x^2 + f_t^2 + f_{xt}^2 + f_{tt}^2\}(x,t)\,dx\,dt \quad (5.14)_2$$

to measure the data, and the existence theorem reads: There is a constant $\mu^* > 0$ such that if

$$U_0^*(u_0, u_1) + F^*(f) \le \mu^* \tag{5.15}$$

then the initial value problem (5.3) (with $B = \mathbb{R}$) has a unique smooth solution u that exists for all $t \ge 0$. Moreover, as $t \to \infty$, u_x, u_t, u_{xx}, u_{xt}, u_{tt} converge to zero uniformly in x. (We note that even if one adds $u_0(x)^2$ to the integrand in $(5.14)_1$ the theorem provides no information on the behavior of the displacement $u(x,t)$ as $t \to \infty$.)

The theorems quoted above are a combination of the results of Dafermos and Nohel (1981), Hrusa and Nohel (1985), Hrusa and Renardy (1986), and Hrusa (1988). The first global existence result for (5.3) was established by MacCamy (1977) in the special case when $\psi = \varphi$. Existence theorems for this special case were also obtained by Dafermos and Nohel (1979) and Staffans (1980). Global estimates for (5.2) with $m(t) = \exp(-\alpha t)$ were obtained by Greenberg (1977).

The results described above can be extended to the constitutive relation (1.17) provided that m satisfies (5.1) and (5.4). In place of (5.5) it should be assumed that

$$H(0,0) = 0, \quad (5.16)_1$$

$$H_{,1}(0,0) > 0, \quad H_{,2}(0,0) < 0, \quad H_{,1}(0,0) + H_{,2}(0,0) > 0. \quad (5.16)_2$$

For memory functions that are smooth on $[0, \infty)$, a more general class of constitutive relations is discussed in Hrusa (1983,1985). The case of an integrable singularity in m is discussed in Hrusa and Renardy (1988).

Renardy (1988) has established a global existence theorem for the constitutive relation (1.16) when m has a nonintegrable singularity at zero. He considered the problem

$$\begin{aligned} u_{tt}(x,t) &= F(u_x(x,t))_x + \int_{-\infty}^t m(t-\tau) M(u_x(x,t), u_x(x,\tau))_x d\tau \\ &+ f(x,t), \qquad 0 \le x \le 1, \quad t \ge 0, \end{aligned} \quad (5.17)_1$$

$$u(0,t) = u(1,t) = 0, \quad t \geq 0, \tag{5.17$_2$}$$

$$u(x,t) = w(x,t), \quad 0 \leq x \leq 1, \quad t < 0, \tag{5.17$_3$}$$

$$u(x,0^+) = w(x,0^-), u_t(x,0^+) = w_t(x,0^-), \quad 0 \leq x \leq 1, \tag{5.17$_4$}$$

where w is a given smooth function that satisfies equation $(5.17)_1$ and the boundary conditions $(5.17)_2$ for $t \leq 0$. Of course, one can always make the history w satisfy the equation of motion $(5.17)_1$ for $t \leq 0$ by defining $f(x,t)$ appropriately for $t \leq 0$. However, it is assumed that f is smooth across $t = 0$ and this represents compatibility of the solution for $t > 0$ with the given history. (We note that when m has a nonintegrable singularity, it is essential to require $w(x,0^-) = u(x,0^+)$.)

The assumptions on F and M are

$$F(0) = 0, \quad M(p,p) = 0 \quad \forall p \in \mathbb{R}, \tag{5.18$_1$}$$

$$F'(0) > 0, \quad M_{,1}(p,p) = -M_{,2}(p,p) > 0 \quad \forall p \in \mathbb{R}. \tag{5.18$_2$}$$

The memory function m is assumed to satisfy

$$m(t) > 0, \quad m'(t) < 0 \quad \forall t > 0, \tag{5.19$_1$}$$

$$\int_0^\infty tm(t)dt < \infty, \tag{5.19$_2$}$$

and an additional condition of frequency domain type. To state this condition let us put

$$K(t) = \int_t^\infty m(s)ds \quad \forall t > 0, \tag{5.20$_1$}$$

$$\hat{K}(i\omega) = \int_0^\infty e^{-i\omega t}K(t)dt \quad \forall \omega \in \mathbb{R}. \tag{5.20$_2$}$$

Finally, we assume there is a constant $C > 0$ such that

$$|Re\hat{K}(i\omega)| \geq C|Im\hat{K}(i\omega)| \quad \forall \omega \in \mathbb{R}. \tag{5.21}$$

(This assumption will be satisfied if, for example, $m(t) \sim t^{-\alpha}$ as $t \to 0^+$ for some α with $1 < \alpha < 2$.) Under the above assumptions, there is a constant $\nu > 0$ such that if f is smooth on $[0,1] \times \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \int_0^1 \{f^2 + f_t^2 + f_{tt}^2\}(x,t)\,dx\,dt \leq \nu \tag{5.22}$$

then the problem (5.17) has a unique solution that is smooth and small (in an appropriate function space) and exists for all $t \geq 0$. Moreover, as $t \to \infty$, u, u_x, u_t converge to zero uniformly in x.

The assumption that f is small (in the sense of (5.22)) implies that the given history w must be small. It is not known whether global solutions exist for large data. This problem is currently under investigation. A local (in time) existence theorem for large data is also given in Renardy (1988). A related local existence theorem is given in Hrusa and Renardy (1988).

6. Remarks on Weak Solutions

By a weak solution we mean a displacement field u that is allowed to have discontinuities in the strain u_x and the velocity u_t, and that satisfies the equation of motion in the sense of distributions. Motions that are well modelled by discontinuous strain and velocity fields are observed experimentally (cf., e.g. Nunziato, Walsh, Schuler and Barker (1974) and Walsh (1985)). Moreover, in view of the results concerning formation of singularities, one must allow for weak solutions (at least when $m(0)$ is finite) if global existence for large data is desired.

There have been many formal studies of shock waves in viscoelastic media (cf. the encyclopedia article of Chen (1973) and the references cited therein). However, despite the importance of the subject, comparatively little is known about the existence of weak solutions.

For a very general class of constitutive relations (with smooth memory functions), Greenberg (1967) established the existence of a special class of weak solutions with steady shocks, i.e. solutions of the form $u(x,t) = g(x + ct)$, where c is a constant and g' has a jump discontinuity. It was pointed out in Section II.6 of RHN that, under appropriate assumptions on H, Greenberg's proof can be applied to the constitutive relation (1.17) when m has an integrable singularity.

For memory functions with integrable singularities Londen (1978) and Engler (1987) have established global (in time) existence of weak solutions. Their result apply to very general classes of data (which permit the initial strain and the initial velocity to be discontinuous); no size restrictions on the data are needed. However, growth conditions (for large strains) are required for the constitutive functions; experimental data indicate that such growth conditions may not always be appropriate. Londen considers the initial-boundary value problem (5.3), (5.7) (with $B = [0,1]$) in the special case when $\psi = \varphi$. Engler studies the constitutive relation (1.17). He also studies certain anti-plane shearing motions of K-BKZ materials that can be regarded as three-dimensional analogues of (1.17); a restriction on the constitutive relation is needed to make such motions possible. The arguments of Londen and Engler make essential use of the assumption that $m(0) = +\infty$.

Recently, Nohel, Rogers, and Tzavaras (1988) established the (global in time) existence of weak solutions to the initial value problem (5.3) with $B = \mathbb{R}$ in the special case when $\psi \equiv \varphi$. They assume that m is smooth on $[0,\infty)$, $\varphi' > 0$, and that φ has exactly one inflection point. Their theorem applies to a very general class of initial data (which permits the initial strain and the initial velocity to be discontinuous). The proof makes essential use of the assumptions that m is smooth on $[0,\infty)$ and that $\psi \equiv \varphi$; the extension to a more general class of constitutive equations does not appear to be straightforward.

The problem of developing a more complete existence theory for weak solutions is currently under active study. As a preliminary step, several authors have studied weak solutions of (scalar) first-order integrodifferential equations (cf., e.g. Greenberg, Hsiao, and MacCamy (1981) and Dafermos (1988)).

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